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A THRESHOLD CRITERION  
FOR PHASE LOCKED LOOP DESIGN  
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A THRESHOLD CRITERION  
FOR  
PHASE LOCKED LOOP DESIGN

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William F. Roland



A THRESHOLD CRITERION

FOR

PHASE LOCKED LOOP DESIGN

by

William F. Roland  
Lieutenant, United States Coast Guard

Submitted in partial fulfillment of  
the requirements for the degree of

MASTER OF SCIENCE  
IN  
ENGINEERING ELECTRONICS

United States Naval Postgraduate School  
Monterey, California

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1963

Roland, W

Thesis

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## ABSTRACT

In modern communications systems where it is necessary to obtain maximum utilization of signal power, the phase locked loop is being designed into the system as a coherent detector. The phase locked loop has a threshold which occurs at very low signal to noise ratios, placing a lower bound on the range of useful signal to noise ratios. Since the loop operation becomes non-linear during threshold conditions, it is difficult to predict the threshold of a given loop during design stages.

This paper presents experimental threshold data normalized to apply to a class of phase locked loops. A new definition of the optimum phase locked loop, and threshold are given, and a design method based on these definitions is described. In conjunction with this, the experimental set-up and method of measurement is described. The instrumentation is interesting because of the previously unexplored measurement technique.

This work was done at Philco Western Development Laboratories, Palo Alto, California. The author wishes to express his appreciation for the help and encouragement of the people in the Communication Sciences Department there. Particular thanks to J. Huylar for his skillful construction of the necessary equipment.



## TABLE OF CONTENTS

Section	Title	Page
1.	Introduction	1
2.	Loop Operation and Analysis	4
3.	Optimum Design Problem	14
4.	Design Method	21
5.	Measurement of Design Data	27
6.	Conclusions	47
7.	Bibliography	48
	Appendix A	50



## LIST OF ILLUSTRATIONS

Figure		Page
2-1	Block Diagram of a Phase Locked Loop	4
2-2	Block Diagram Showing Use of Notation	6
2-3	Linear Equivalent Phase Locked Loop	8
2-4	Signal Plus Noise Phasor Diagram	8
2-5	Phase Plane Trajectories of a Phase Locked Loop	12
3-1	Effect of Non-Linear Operation on Output Signal Power	17
5-1	Block Diagram of Equipment set-up	32
5-2	RMS Phase Error vs Signal to Noise Ratio	33
5-3	ENCS vs RMS Phase Error	34
5-4	Doppler Rate Threshold	35
5-5	Increase of ENCS vs RMS phase Error	36
5-6	Normalized Peak Phase Error vs Damping	37
5-7	Oscilloscope Photos Loop Performance	40
5-8	Oscilloscope Photos Loop Performance	41
5-9	Oscilloscope Photos Loop Performance	42
5-10	Oscilloscope Photos Loop Performance	43
5-11	Oscilloscope Photos Loop Performance	44
5-12	Oscilloscope Photos Loop Performance	45
5-13	Oscilloscope Photos Loop Performance	46



TABLE OF SYMBOLS AND ABBREVIATIONS

Symbol	Definition
S	received signal power
$\Delta\omega$	magnitude of step change of received frequency
$N_o$	one sided power spectral density of input noise
$\Delta f_{if}$	effective noise bandwidth of input to loop
S/N	input signal to noise ratio
$s(t)$	input signal, phase or frequency modulated
$s(t)$	receiver's estimate of $s(t)$
$\theta(t)$	phase modulation of $s(t)$
$\theta(t)$	receiver's estimate of phase of $s(t)$
$\phi(t)$	error of $\theta(t)$ in radians
$e(t)$	error voltage, equals $\sin\phi(t)$
$n(t)$	noise voltage input to phase locked loop
$\omega_n$	natural frequency of the linearized loop
$\zeta$	damping factor of the linearized loop
K	loop gain of the linearized loop ( $= \omega_n^2$ )
$k_m$	multiplier efficiency
$k_v$	conversion gain of voltage controlled oscillator
$G(s)$	transfer function of the loop filter
$H(s)$	transfer function of the loop, varies depending on the point at which the output is taken
VCO	voltage controlled oscillator
ENCS	expected number of cycles slip



## 1. Introduction.

The phase locked loop is a coherent detection device for use in receiving radio signals. Because it is a coherent detector, it is able to detect signals at very low signal to noise ratios. It has been put to use in many systems where incoherent detectors, such as the diode detector would be useless. Satellite communication, tracking and telemetry receivers, and very low frequency frequency standards receivers are excellent examples of the possible use of the phase locked loop.

Phase locked loops have an inherent non-linearity which places a limit on the noise levels in which they will work. This limit of performance is called a threshold, and is a function of many parameters, some of which are under the control of the designer. However, the mathematical tools are not yet available to determine analytically the optimum parameters to minimize the loop threshold in a given environment.

Many authors have approached phase locked loop analysis in an attempt to predict performance. The earliest work utilized entirely linear analysis. Jaffe and Rechtin (7), Gilchrist (6), Weaver (8), and Gruen (19) have each presented excellent linear analyses of phase locked loops, and have shown optimization procedures.

More recently, work has been directed toward solving analytically



for phase locked loop performance when the non-linearity is carried into the calculations. Viterbi (9) has written a very complete report on analog computer studies and on analytic studies of the complete loop performance. Cahn (13) has completed an analytic solution to phase locked loop performance, directed toward the acquisition problem, utilizing a piecewise linear approximation of the dynamic characteristics of the loop.

Many of the latest studies have been attempts at predicting the dynamic performance of the phase locked loop in noise when the nonlinearities are included in the analysis. The difficulty comes not only from the lack of tools to solve the differential equations, but also because the probability density function associated with phase error becomes unmanageable mathematically. Tikhonov (10,11) has recently developed the mathematical tools to find the joint probability density function for phase error and frequency error, but it has not been shown how this may be used in design. Develet (1) has derived threshold curves for the situation in which the phase error becomes unbounded, using Boonton's quasi-linearization technique. Van Trees (4) has developed a similar threshold equation, without the linearizing approximation. Both of these developments lack the generality that is needed in design work where many parameters effect the threshold and must be accounted for.



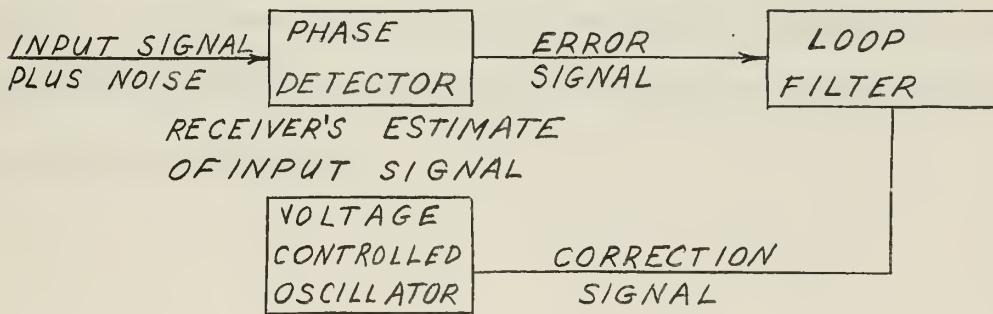
It is the intent of this paper to analyze phase locked loop performance, mathematically where possible and experimentally where mathematical techniques are not available, and to show a method of phase locked loop design. This method will include precise definitions of "optimum" and "threshold" and will allow the designer to predict loop performance when phase errors due to noise and modulation transients cause the loop to operate non-linearly. The design is intended to optimize the phase locked loop's ability to pull a signal out of the noise.



## 2. Loop operation and analysis.

The phase locked loop operates as a cross-correlation or coherent detector. The incoming signal is multiplied by the receiver's estimate of the signal. The correlation function is a sinusoid as a function of the phase error between the received signal and the estimate, having zeros at  $\pm 90^\circ$  phase error. The multiplication operation described above results in a voltage equal to the value of the correlation function plus double frequency terms which are filtered out. The correlation voltage is applied so as to change the receiver's estimate of the incoming signal and drive the phase error to  $+90^\circ$ .

Figure 2-1 is a block diagram of a typical phase locked loop. The phase detector performs the multiplication adequately for the instrumentation of a physical loop. The loop filter may be designed under many optimization criteria to obtain best loop performance. The voltage controlled oscillator output is the receiver's estimate of the incoming signal shifted in phase by  $90^\circ$ .



Phase Locked Loop  
Figure 2-1



The loop output may be taken at various points depending on the modulation technique. For phase modulation, the output is taken at the phase detector output. For frequency modulation the output is taken at the loop filter output. If the phase locked loop is to be used as a narrow band filter the output is taken at the VCO. Amplitude modulation of the received signal may be detected by using the phase locked loop as a narrow band filter to isolate the carrier. Then the received signal is multiplied by the VCO output phase shifted  $90^\circ$ . The output of this multiplier is the modulation waveform. This technique will also give an indication of the presence of a carrier, regardless of modulation.

The loop may be analyzed by redrawing the loop, using the notation as in figure 2-2. Note that the voltage controlled oscillator is considered a perfect integrator. The oscillator output is a constant amplitude sine wave, that is a voltage with phase increasing as a function of time, and the frequency, or phase rate, is a function of the input. Since phase rate is the time derivative of phase, the voltage controlled oscillator is an integrator with a gain constant having units of radians per volt sec.



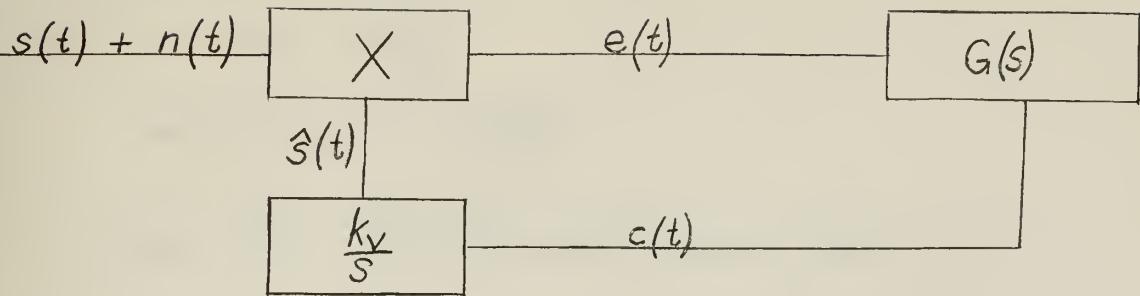


Figure 2-2

To analyze the loop, consider  $s(t) = A \sin(\omega t + \theta(t))$ , where the information is in  $\theta(t)$ , and  $\hat{s}(t) = C \cos(\omega t + \hat{\theta}(t))$ . The phase detector output is:

$$e(t) = s(t)\hat{s}(t) = \frac{AC}{2} [\sin(\theta(t) - \hat{\theta}(t)) + \sin(2\omega t + \theta(t) + \hat{\theta}(t))]$$

The double frequency term will not be considered further as it is to be completely filtered out. Then:

$$e(t) = \frac{k_m A C}{2} \sin(\theta(t) - \hat{\theta}(t)); k_m = \text{multiplier efficiency}$$

Define:  $\phi(t) = \theta(t) - \hat{\theta}(t)$ ,

$$\text{and } c(t) = \mathcal{L}^{-1} \left[ G(s) \mathcal{L} \left( \frac{k_m A C}{2} \sin \phi(t) \right) \right] = \mathcal{L}^{-1} \left[ G(s) E(s) \right].$$

$$\text{Then } \ddot{\phi}(t) = \ddot{\theta}(t) - \ddot{\hat{\theta}}(t) = \ddot{\theta}(t) - \dot{c}(t)k_v,$$

where  $k_v$  is the vco gain in radians per volt sec.

$$\text{Now } \ddot{\phi}(t) = -\mathcal{L}^{-1} \left[ s G(s) \mathcal{L} \left( \frac{k_m A C k_v}{2} \sin \phi(t) \right) \right] + \ddot{\theta}(t).$$

Jaffe and Rechtin have derived the optimum form for  $G(s)$  under the



Weiner criterion to be:

$$G(s) = \frac{\tau_1 s + 1}{\tau_2 s} .$$

If we also define  $K = \frac{k_m A C_v}{2\tau_2}$ ,

$$\begin{aligned} \text{then } \ddot{\phi}(t) &= -\mathcal{L} \left[ \frac{\tau_1 s + 1}{s} (K) \sin \phi(t) \right] + \ddot{\theta}(t) \\ &= -K \left[ \tau_1 \dot{\phi}(t) \cos \phi(t) + \sin \phi(t) \right] + \ddot{\theta}(t) . \end{aligned}$$

The solution of this non-linear differential equation is not known.

However, if the linearizing assumption is made that  $\cos \phi(t) \approx 1$

and  $\sin \phi(t) \approx \phi(t)$ , then the loop equation may be written:

$$\ddot{\phi}(t) = -K\tau_1 \dot{\phi}(t) - K\phi(t) + \ddot{\theta}(t) ,$$

or in Laplace notation:

$$(s^2 + K\tau_1 s + K) \phi(s) = s^2 \theta(s) ,$$

and

$$\frac{\phi(s)}{\theta(s)} = \frac{s^2}{s^2 + K\tau_1 s + K} .$$

If  $C(s)$  is the output then:

$$\frac{C(s)}{\theta(s)} = \frac{s(\tau_1 s + 1)}{\tau_2 (s^2 + K\tau_1 s + K)} ,$$

or if  $\hat{\theta}(s)$  is the output:

$$\frac{\hat{\theta}(s)}{\theta(s)} = \frac{k_y(\tau_1 s + 1)}{\tau_2 (s^2 + K\tau_1 s + K)} .$$

Using standard servo notation:

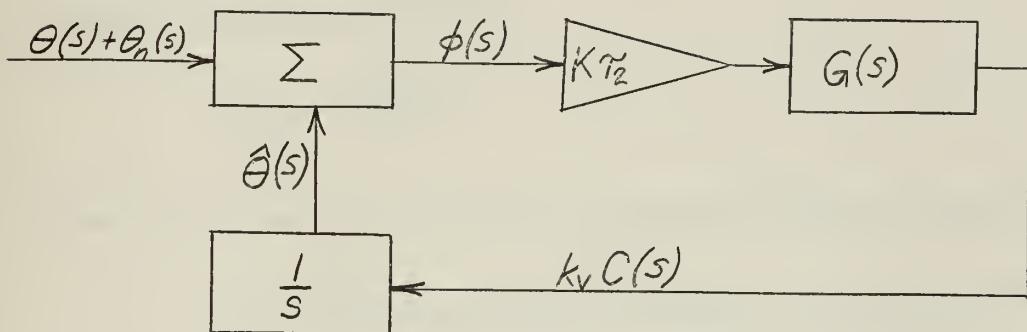


$$\omega_n = \sqrt{K} = \text{natural frequency of the loop,}$$

and

$$\zeta = \frac{\tau_1 \sqrt{K}}{2} = \text{damping factor of the loop.}$$

The linearized model derived above may be redrawn as shown in figure 2-3.



Linear equivalent phase locked loop  
Figure 2-3

Noise may be introduced into the linear model in the following manner. First consider the input noise to be bandpass gaussian noise. Then it can be shown that the noise waveform may be expressed as:

$$n(t) = n_q(t) \cos \omega t + n_i(t) \sin \omega t$$

Where  $n_q(t)$  and  $n_i(t)$  are gaussian random variables having the same standard deviation as  $n(t)$ . Then from figure 2-4 it is obvious that the phase jitter on a sinusoid to which this noise is added is:

$$\theta_n(t) \cong \frac{n_q(t)}{A}$$

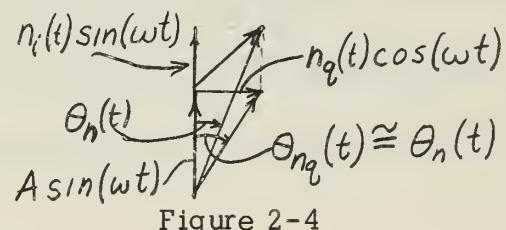


Figure 2-4



Now, noting that:

$\sigma_n^2 = \sigma_{n_q}^2 = N_0 \text{BW}$ ; where  $N_0$  = the one sided power spectral density of the noise.  
BW = input noise bandwidth

Then:  $\sigma_{\theta_n}^2 = \frac{\sigma_n^2}{A^2} = \frac{N_0 \text{BW}}{A^2}$

and the one sided power spectral density of the low pass phase noise having a bandwidth  $\frac{\text{BW}}{2}$  is:

$$G_{\theta_n}(j\omega) = \frac{N_0 \cdot 2}{A^2}$$

The total phase noise power in the estimate  $\hat{\theta}(t)$  may now be found:

$$P_{\theta_n} = \int_0^{+\infty} G_{\theta_n}(j\omega) |H(j\omega)|^2 d\omega; H(j\omega) = \text{loop transfer function}$$

= variance of  $\theta$

Using this type of development for noise, Jaffe and Rechtin (7) applied the Wiener minimum mean square error criterion to the linearized phase locked loop to find the optimum loop. Their results showed that for an input signal modulation of a step in frequency, the optimum phase locked loop has a bandpass limiter at the input, and integral plus proportional loop filter, and a damping factor of 0.7. Their development did not show any optimum value for the natural frequency or bandwidth of the loop. Later work by Gilchrist (6) supports these findings and generalizes on them, but still does not assign an optimum



value for the loop bandwidth. Appendix A contains a derivation of the optimum values of parameters for the loop using Liapunov's second method. Since the choice of Liapunov function is arbitrary, there is no real guarantee that the values indicated are truly optimum. Nevertheless Liapunov's method is not completely without merit, since it allows the designer complete freedom in the choice of optimization criteria. Rather than minimum mean square error, the designer may choose to minimize the settling time of the loop under transient conditions, or utilize the "integral time, absolute error" criterion.

Note in all of these optimization efforts that the linear equivalent phase locked loop is utilized. This is significant because it is not in this mode of operation that difficulty is experienced in the use of a loop. The problems come when phase errors are large and the linearizing assumption is no longer valid.

If we go back now and look at the complete non-linear equations for the phase locked loop, we can obtain some information on the dynamic performance from a plot of phase plane trajectories. (Note that the phase plane is not directly related to the phase angle under discussion, but is a tool used in non-linear servo analysis.)

The dynamic state of the phase locked loop may be completely described if the phase error  $\phi(t)$  and phase error rate  $\dot{\phi}(t)$  are known.



A two dimensional plot of  $\dot{\phi}(t)$  vs  $\phi(t)$  is called a phase plane. From any given point on this phase plane, there is one and only one trajectory or locus of points through which  $\dot{\phi}(t)$  and  $\phi(t)$  may vary as these quantities change toward their stable values.

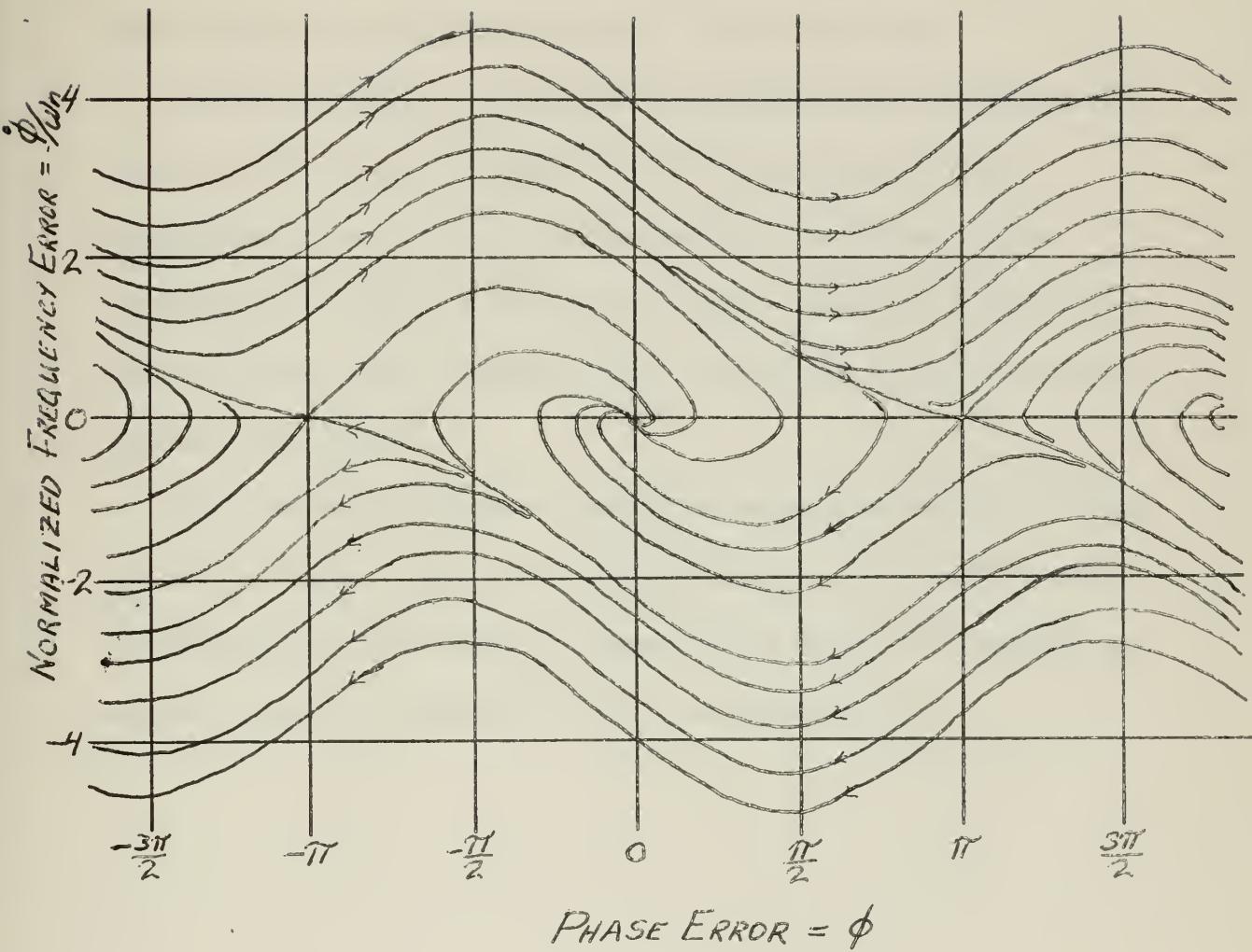
Phase plane trajectories may be plotted analytically by determining the slope of the trajectories at each point in the phase plane from the differential equations of the loop. The slope at any point in the phase plane is:

$$\begin{aligned}\frac{d(\dot{\phi}(t))}{d\phi(t)} &= \frac{d(\dot{\phi}(t))/dt}{d\phi(t)/dt} = \frac{\ddot{\phi}(t)}{\dot{\phi}(t)} \\ &= \frac{-K\tau_1\dot{\phi}(t)\cos\phi(t) - K\sin\phi(t)}{\dot{\phi}(t)} \\ &= -K\tau_1\cos\phi(t) - \frac{K}{\dot{\phi}(t)}\sin\phi(t)\end{aligned}$$

From this equation, for each value of  $\phi(t)$  and  $\dot{\phi}(t)$ , the slope of the trajectory at that point may be plotted. Then trajectories may be plotted as in figure 2-5.

The experimental determination of phase plane trajectories has been well described in work by Viterbi (9) and Rue and Lux (12). Viterbi presents phase plane trajectories for a great variety of first, second and third order phase locked loops. Order number is the number of integrators in the loop. These phase plane diagrams also show the effects of various types of modulation. He has derived analytically





Phase Plane Trajectories of a Phase Locked Loop  
Figure 2-5

the approximate acquisition range and time for these various loops.

Acquisition range is the number of cycles per second the VCO can be pulled away from its center frequency to phase lock on a signal with which it was not previously in lock. This is different from the tracking



range which is the frequency change over which a signal which is in lock can pull the VCO. These developments are for the case of infinite signal to noise ratio at the phase locked loop input.

Frazier and Page (15) have done experimental work on determining the acquisition range of a phase locked loop in noise. This is a compilation of data from many runs which show the probability of acquisition of a signal in noise if the voltage controlled oscillator is swept past the signal at various rates. Acquisition is the only problem treated in this paper.

This discussion of phase locked loop analysis has been sufficient to give the reader a grasp of the design problem. For a more detailed understanding of the analysis of phase locked loops, he is referred to the many excellent treatments listed in the bibliography.



### 3. Optimum Design Problem.

Optimization of the design of any system involves the adjustment of equipment configuration and parameter values to perform the intended task in the best way. There are four main steps to be carried out:

- 1) Determine the problem. For phase locked loop design this is to demodulate or remove information from a signal in the presence of noise. The type of modulation and the minimum expected signal to noise ratio must be known.
- 2) Identify those parameters which may be varied to obtain the best performance. These are natural frequency of the loop, loop damping factor, configuration of the loop filter, and the use or non-use of a bandpass limiter at the loop input.
- 3) Establish a measure of effectiveness or the criterion to be maximized or minimized by varying the parameters.
- 4) Compare the performance of the loop by this criterion and choose those values of the parameters which best satisfy the criterion.

The first step is quite easily carried out. The information may be given either explicitly or implicitly in the specifications for the loop. The next step is also easily disposed of since these parameters are listed and do not change from one design problem to the next. The three most common types of loop filter are the integral plus propor-



tional filter  $\frac{\tau_1 s + 1}{\tau_2 s}$ , the lead-lag RC filter  $\frac{\tau_1 s + 1}{\tau_2 s + 1}$ , and the simple RC low pass filter  $\frac{1}{\tau_1 s + 1}$ .

Steps three and four are intimately related and require considerably more thought than the first two. There are as many possible criteria for optimization as there are designers in the field. The Wiener minimum mean square error criterion has gained general acceptance as the best criterion for optimizing linear systems. However in the field of non-linear systems there is no generally accepted optimization criterion.

Jaffe and Rechtin (7) have carried out this optimization procedure for the linear equivalent phase locked loop. They chose the Wiener minimum mean square error criterion as the measure of effectiveness, and the signal modulation considered was a step of frequency. Their results showed that the optimum phase locked loop has a bandpass limiter at the loop input, the loop filter is an integral plus proportional filter  $(\frac{\tau_1 s + 1}{\tau_2 s})$ , and the loop damping factor is 0.7. The natural frequency of the loop or the loop bandwidth is not specified in their work, nor does the Wiener criterion give an optimum value for this parameter. Gilchrist (6) in more recent work concurred with Jaffe and Rechtin, and generalized the result. However, to date there has been no optimum bandwidth criterion defined.

It was pointed out in the introduction that one of the most important



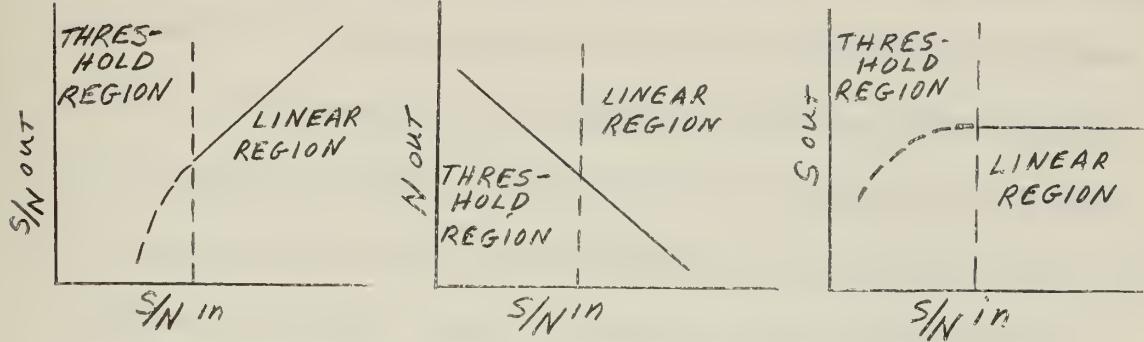
features of the phase locked loop is its ability to operate at noise levels far below those possible with incoherent detectors. Also it was noted that there is a practical limit or threshold input signal to noise ratio below which a phase locked loop will not operate properly. Because the threshold of a phase locked loop varies with the design parameters and signal modulation and because operation in noise is its strong point, an optimum design of a phase locked loop should maximize its ability to pull a signal out of high noise levels. That is, the optimum phase locked loop for receiving a given signal is that phase locked loop which has the lowest threshold.

This immediately brings up the question of a definition of threshold. The definition must be generally applicable, and it must relate threshold to the output quality requirements of the specifications. Looking first qualitatively at phase locked loop performance near threshold, a number of effects may be observed, any one of which may be used to define threshold.

A) As the input signal to noise ratio decreases, the output signal to noise ratio remains linearly related to it until the threshold region is reached. Then the output signal to noise ratio decreases more rapidly. However, looking a little more closely at this we find that the input and output noise powers remain linearly related, while the



signal power begins to drop off. This is shown graphically in figures 3-1a, b, c. Signal input power is constant in all cases, while noise power is increased to decrease signal to noise ratio.



Threshold Effect on Output S/N  
 Figure 3-1

B) The rms phase error remains linearly related to increasing noise power up to a point, and then increases more rapidly than input noise in the threshold region. If we arbitrarily say that phase error cannot exceed  $180^\circ$  then the rms phase error has a limit of  $104^\circ$ . This is the rms phase error of two uncorrelated sine waves. The arbitrary limit of  $180^\circ$  on the phase error is not unreasonable because, if phase error exceeds  $180^\circ$ , the stable point in the phase plane shifts  $360^\circ$  in the direction of the phase error. Then the phase error no longer exceeds  $180^\circ$ .

C) Another effect related to rms phase error is cycle slipping. This is the effect observed when the stable point in the phase plane shifts  $360^\circ$ , and is equivalent to the VCO phase changing  $360^\circ$



relative to the incoming signal. Cycle slipping does not occur outside of the threshold region, but occurs at an increasing average rate as the noise level increases in the threshold region. It is a random event associated with the noise peaks which cause phase errors in excess of  $180^{\circ}$ , and is aggravated by transients propagated around the loop by the signal modulation. The time that the phase locked loop takes in changing stable points is approximately the reciprocal of the natural frequency of the loop. There is therefore a limit to the number of cycles slip possible in a given period of time.

Van Trees (4) and Develet (1) have each found an analytical method for finding that input signal to noise ratio at which the rms phase error becomes unbounded and have proposed that this be used as the definition of phase locked loop threshold. However, this will have certain difficulties, particularly if it is used in design work. The signal to noise ratio at which the phase error becomes unbounded is an absolute minimum for the operation of a phase locked loop. In general the loop output will become useless in terms of the system specifications of output quality at some signal to noise ratio higher than that given by this threshold. In addition Develet (1) has been able to predict analytically the output signal to noise ratio of the phase locked loop during non-linear operation, for a certain class of



modulation. He has defined threshold as the input signal to noise ratio for which the output signal to noise ratio reaches a certain minimum, the minimum usually being specified by system requirements.

However, since the designer is not always interested in output signal to noise ratio, but may be interested in rms phase error, or output distortion, or frequency accuracy, the threshold definitions given above do not have the necessary generality. A threshold criterion in terms of cycle slipping has many interesting properties and can be easily made quite general. The expected number of cycles slip in any given period of time is directly related to the probability of the phase error exceeding  $180^\circ$ .

Cycle slipping can be used to explain effects A and B above. If we consider that output signal is blanked during cycle slipping and compute the power reduction of the signal due to slipping, the signal output power is predictable to within 2 db of the measured signal power reduction. Cycle slipping also explains the slight increase of rms phase error over that expected from linear considerations, because a transient is propagated through the loop when cycle slipping occurs.

Cycle slipping may also be related to the accuracy of frequency measurement when the phase locked loop is used for doppler tracking



of satellites. Since cycle slipping causes, effectively, an extra count or shortage of a count at the frequency determining device, the maximum error of count is the expected number of cycles slip during the count period.

Threshold is therefore defined as that input signal to noise ratio for which cycle slipping occurs at a specified maximum rate. This maximum rate is determined for each phase locked loop by the designer according to the requirements on the quality of the output as stated in the system specifications.



#### 4. Design Method

The optimum phase locked loop has been defined as that phase locked loop having the lowest threshold under a given set of conditions of noise and modulation. Where threshold is defined as that input signal to noise ratio for which the expected number of cycles slip per second reaches a minimum. This is equivalent to saying the optimum loop is that loop which, for the given signal modulation and noise conditions, has the minimum expected number of cycles slip per second.

The designer might try any one of three possible methods of finding the expected number of cycles slip per second.

He might try to determine the probability density function of phase error and in that way predict the expected number of cycles slip per second. However since the mathematical tools are not available for working with non-gaussian random variables, this approach comes quickly to a dead end.

Another approach, which is the most accurate but is hardly sophisticated, is to construct and test a number of phase locked loops to find the optimum one. However, this becomes too expensive if many loops are to be built.

The third method, and one which requires general acceptance of one criterion for phase locked loop design, is the use of empirical design data. This requires that the optimum phase locked loop be precisely defined and that the optimization criterion be useful in many applications.



Cycle slipping meets the optimization criterion requirements nicely.

This is most easily seen if the optimization is first considered in terms of finding optimum loop bandwidth only. As the loop bandwidth is decreased the amount of noise power which is within the loop passband becomes less, and the probability of a noise peak causing cycle slipping decreases. However, as bandwidth decreases the transient phase errors due to modulation increase in magnitude. These transient peaks added to the noise cause the probability of cycle slipping to increase. If transient errors become greater than  $180^\circ$ , they will cause cycle slipping without noise. Therefore, there is some optimum bandwidth which minimizes the expected number of cycles per unit time.

As an example consider the case of receiving a fixed carrier with no modulation, or doppler, such as the signal from a VLF standard frequency transmitter. The first assumption is that the passband of the receiver should be as narrow as it is possible to make it. But we will find that it will not work. If the loop had zero bandwidth, no information would pass through the loop, and the result would be that we would not know if the loop was phase locked to the carrier. As a matter of fact, the loop would not be in lock because the carrier is phase modulated by the variations of the height of the ionosphere. This is essentially information on the carrier which the phase locked loop must estimate to be phase locked. So there is minimum bandwidth at which the phase locked loop will operate, and some greater bandwidth at which the variance of the



phase error will be minimum.

Or if we look at the case of an intentionally modulated signal at the input, for instance a frequency shift keying type of signal, the input is effectively a ramp of phase. There is a transient peak of phase error resulting from each change of frequency and the narrower the loop bandwidth the greater will be the peak value of the phase error. During the time this peak phase error exists, a noise peak may be added to it and cause a cycle to slip where the noise alone would not have caused slip. The same is true of all types of modulation.

It can be seen from this discussion that there is an optimum loop bandwidth which will result in the minimum number of cycles slip. To find this optimum bandwidth from empirical data requires that the expected number of cycles slip be recorded for many types of signals, signal bandwidths, and loop filters with many different signal to noise ratios. The magnitude of the data can be reduced by breaking it into two parts and by normalizing.

The normalized input signal to noise ratio is defined as the signal to noise ratio in twice the loop bandwidth ( $S/2f_n N_o$ ). Where  $S$  is the signal power,  $f_n$  is the natural frequency of the loop, and  $N_o$  is the one sided power spectral density of the noise at the loop input. The normalized expected number of cycles slip per second is the expected number of cycles slip per bandwidth time period ( $1/f_n$ ). This is justified by considering that if some arbitrary phase error is put into the loop, it will return



to the stable condition at some rate which is determined by the natural frequency of the loop. Therefore the maximum rate at which cycle slipping can occur is related to the natural frequency. The actual number of cycles slip per  $1/f_n$  will then be some number less than one, and will increase as the noise power increases.

The data may be separated in two parts by considering first only the expected number of cycles slip per  $1/f_n$ , and then looking at the increase in this number due to various types of modulation. First data would be taken on a phase locked loop for the expected number of cycles slip per  $1/f_n$ , over a range of normalized input signal to noise ratios. This data should then be applicable to any phase locked loop with the same loop filter type and the same condition of input limiting.

Then data would be taken on the increase of expected number of cycles slip per peak of phase error. For a particular signal modulation, the number of peaks per  $1/f_n$  will be known, and the increase of cycles slip per  $1/f_n$  may easily be found from this data. The signal types for which data must be taken are: step of frequency; ramp of frequency; and the worst case of analog modulation which is sinusoidal modulation at modulation frequency equal to the loop natural frequency.

Now the designer can use this data to find the optimum bandwidth for a phase locked loop having the same loop filter and input circuit as used in obtaining the data. Knowing the minimum received signal power and maximum noise power spectral density for his design problem, he can



replot the data of the first curve as expected number of cycles slip per second versus loop natural frequency.

$$f_n = \frac{1}{2} \frac{S/N_0}{S/2f_n N_0}$$

$$\text{ENCS/sec} = \text{ENCS/} \frac{1}{f_n} \times f_n$$

Then add to this curve the increase of expected number of cycles slip per second due to transient peaks. Since the form of the received signal is known, the number of transient peaks per second will be known. The transient peak level must be found from empirical data. This peak level will vary with bandwidth. For each value of  $f_n$  find the peak of phase error, and normalized signal to noise ratio corresponding. From the second part of the data, find the increase of expected number of cycles slip per transient and convert to increase of expected number of cycles slip per second.

If the increase is added to the first curve at corresponding values of  $f_n$ , the resultant curve will have a minimum point at the optimum bandwidth.

This design procedure has optimized the loop for only one parameter. The design must be repeated for each combination of loop filter and input filter (i.e., with and without limiting). The only remaining parameter is damping factor. If the discussion is limited to damping factors between 0.5 and 1.0, the effect of changing the damping may be easily observed. There are two major effects: the effective noise bandwidth of the loop is minimum at damping factor equals 0.5 and increases about 1 db as damping factor is increased to 1.0. However, the peak overshoot is



maximum when the damping factor is 0.5 and decreases as the damping increases. If the design data were repeated for many values of damping factor it would become unwieldy, and unnecessarily so, since Jaffe and Rechtin (7) have shown that a damping factor of 0.7 is optimum for the linear system, and the damping factor is undefined for a non-linear system. Therefore it will be assumed that the value of 0.7 for damping factor is optimum.

It should be pointed out to the designer that the assumption that the minimum signal to noise density ratio is known may not be sufficient. There are two possibilities: one that the noise level is so high that even at the optimum bandwidth, the expected number of cycles slip per second is such that the output will not meet the quality requirements of the specifications. In this case the signal cannot be demodulated. The other case is that the noise is so low that there is a range of values of bandwidth over which the expected number of cycles slip per second is unmeasurable. In this case the threshold design criterion does not fill the need for a design method. There are two ways out: One is to find some other criterion, and the other is to assume another arbitrary higher noise level which will permit optimization. The second is the recommended course.

The design data for the integral plus proportional filter and bandpass gaussian noise is presented in the next section. Further data is needed to complete the set of design data.



## 5. Measurement of Design Data.

To measure the phase locked loop performance it is necessary to be able to observe the input signal to noise ratio, peak phase error, and number of cycles slip. In addition equipment was arranged to observe the rms phase error and the phase plane trajectories.

Figure 5-1 is a block diagram of the laboratory set-up of equipment, and is referred to in the following discussion of the equipment.

The phase locked loop is a second order loop. The operation is described in detail in section 2. The center frequency of operation is 500 kc. The phase detector is of the diode chopper type which has the advantage of allowing direct observation of phase error, and has wide dynamic range. The loop filter is of the integral plus proportional type. It uses an operational amplifier to obtain effectively infinite integration time. The loop gain is controlled by varying the input resistor  $R_1$ , and loop damping factor is controlled by  $R_2$ . Damping factor is directly proportional to  $R_2$ , and gain is independent of it. Loop gain or  $\omega_n^2$  is directly proportional to  $R_1$ , but varying  $R_1$  also varies the damping factor. The voltage controlled oscillator is of the multivibrator type with the control voltage varying the base bias of the transistors. By careful design and construction, it was possible to get accurately predictable dynamic performance of the loop, with excellent repeatability of measurements.

The signal source is a laboratory built fm generator capable of modulation from dc to 5 kc. Linearity is within 10% over a range of



$\pm 5$  kc from 500 kc. The modulator is a Hewlett Packard 202 function generator, which is capable of sine, square, and triangular outputs from 0.01 cps to 1200 cps. The possible outputs from the fm generator are therefore sinusoidal frequency modulation, frequency steps and frequency jumps.

The noise source is a General Radio 1290B random noise generator with a 5 mc bandwidth. The noise has essentially gaussian characteristics. A Collins 51J receiver with a 6 kc mechanical bandpass filter is used to isolate one portion of the noise spectrum, increase its amplitude, and to center it at 500 kc. The step attenuator allows precise control of the input signal to noise ratio.

The linear phase comparator is the heart of the measurement method. It has an output voltage which is directly proportional to the instantaneous phase difference of the VCO and the uncorrupted input signal, up to  $\pm 180^\circ$ . The rms phase error may be observed directly on a true rms reading voltmeter. The output may be observed on an oscilloscope to find peak phase errors and to observe phase plane trajectories of the loop operation. Most important to the work here is that when the phase error exceeds  $\pm 180^\circ$ , the output of the phase comparator switches from maximum positive to maximum negative output or visa versa. The bandwidth of this transient is much greater than the noise and modulation bandwidth of the phase error signal. Therefore by using a highpass filter, the transient may be completely removed from the noise, leaving a clean pulse. This filtered transient is a single cycle having one positive and one negative peak, at a frequency of



about 20 kc. It is fed directly to a counter which counts an event each time a pulse is received.

To observe the expected number of cycles slip for the various conditions listed in section 4, the input signal to noise ratio must first be determined. The signal level is first set, without noise, to that value for which the phase locked loop is designed, and the level is observed on the true rms reading voltmeter. Then the signal is removed, the step attenuator set to 0 db and the noise level adjusted to the same reading as the signal level. The attenuator dial now reads directly in signal to noise ratio in a 6 kc bandwidth ( $S/6\text{kc} \times N_o$ ).

To obtain data on the expected number of cycles slip per  $1/f_n$  versus the signal to noise ratio in the loop bandwidth ( $S/2f_n N_o$ ), the input signal to noise ratio is set, and the counter gate opened for a relatively long period of time (T seconds). The expected number of cycles slip (ENCS) per  $1/f_n$  is:

$$\text{ENCS}/\frac{1}{f_n} = \frac{\text{total count } (N_n)}{T \times f_n}$$

and the normalized signal to noise ratio is

$$\frac{S}{2f_n N_o} = \frac{S}{6\text{kc} \times N_o} \times \frac{6\text{kc}}{2f_n}$$

This curve is plotted in figure 5-3. Note that the ordinate is rms phase error, rather than signal to noise ratio. The conversion from rms phase error to signal to noise ratio may be made from figure 5-2.



The increase of the number of cycles slip due to transients in the loop is found by frequency modulating the input signal. The effect of a step of frequency is found by square wave modulating the generator at a frequency much less than the natural frequency of the loop, and observing the peak phase error on the oscilloscope. The peak phase error is set to a particular value by adjusting the magnitude of the frequency step. Then the noise level is increased and the number of cycles slip in a long period of time (T) is counted. Then the increase in cycles slip per peak of phase error is found as follows:

$$\text{increase of ENCS/peak} = \frac{\text{total count } (N_{s+n}) - N_n}{T \times 2f_m}$$

$T \times 2f_m$  = number of peaks of phase error in time T.

This data is presented in figure 5-4 for several values of peak phase error.

The effect of sinusoidal modulation is found by exactly the same technique. However the modulation frequency chosen is the same as the loop bandwidth ( $f_n$ ). As may be seen in figure 5-4, the results correlate quite well with those for the square wave modulation, up to the point where part of the time the loop slips more than one cycle per transient due to the square wave.

With a fixed phase offset, the results are not as might be expected from the previous discussion. In a second order loop a fixed phase offset is due to a fixed rate of change of input frequency. The phase error is:



$$\phi = \sin^{-1} \left( \frac{\dot{\omega}}{\omega_n^2} \right)$$

It was intended that this data measure the increase of expected number of cycles slip per  $1/f_n$  with phase offset. However, it was found that, for fixed phase errors greater than  $10^\circ$ , when the loop slipped a cycle there was a finite probability that it would not recover lock again. Instead it would fall behind the input signal frequency. For fixed phase errors less than  $10^\circ$  the expected number of cycles slip was the same as for  $0^\circ$  offset.

The reason for loss of sync can be seen qualitatively in the phase plane plot for a phase locked loop with a fixed phase offset.<sup>1</sup> The phase plane region of unstable trajectories is close to the stable point and there is only a narrow region in which trajectories allow cycle slip and then return to another stable point.

Therefore, data was taken to measure the probability of loss of sync per  $1/f_n$ , and an additional threshold was defined. Doppler rate threshold is that input signal to noise ratio such that the probability of loss of sync per  $1/f_n$  is  $10^{-6}$ . This threshold is plotted as a function of fixed phase error in figure 5-5.

Figure 5-6 is a plot of normalized peak phase error for various damping factors and normalized frequency steps. This data may be used to

<sup>1</sup> See Viterbi, A. J. (9) pp. 48-51



BLOCK DIAGRAM OF THE EQUIPMENT SETUP

FOR MEASUREMENT OF EXPECTED CYCLES SLIP, RMS PHASE ERROR,

AND PEAK PHASE ERROR

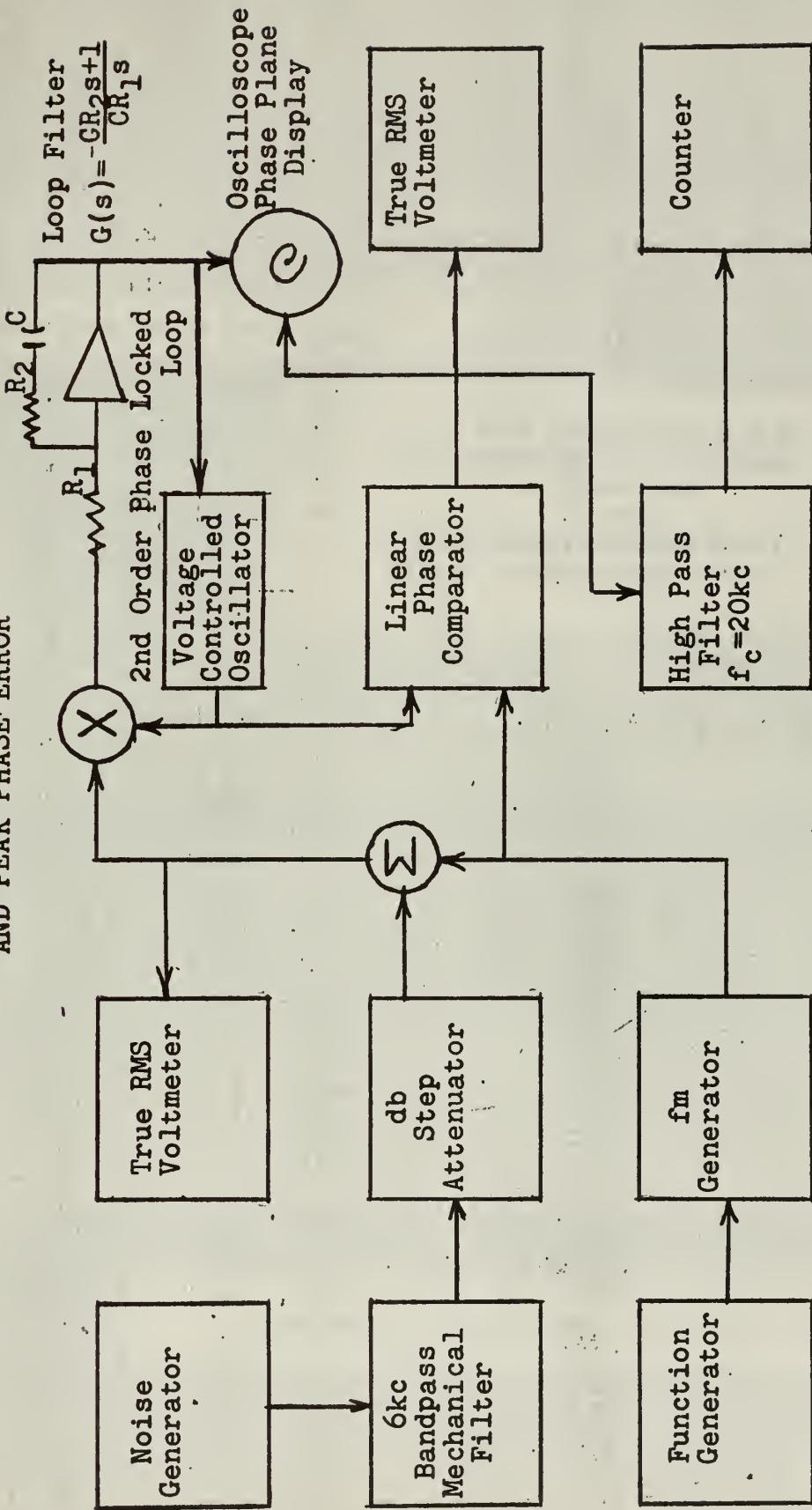


Figure 5-1



FIG. 5-2

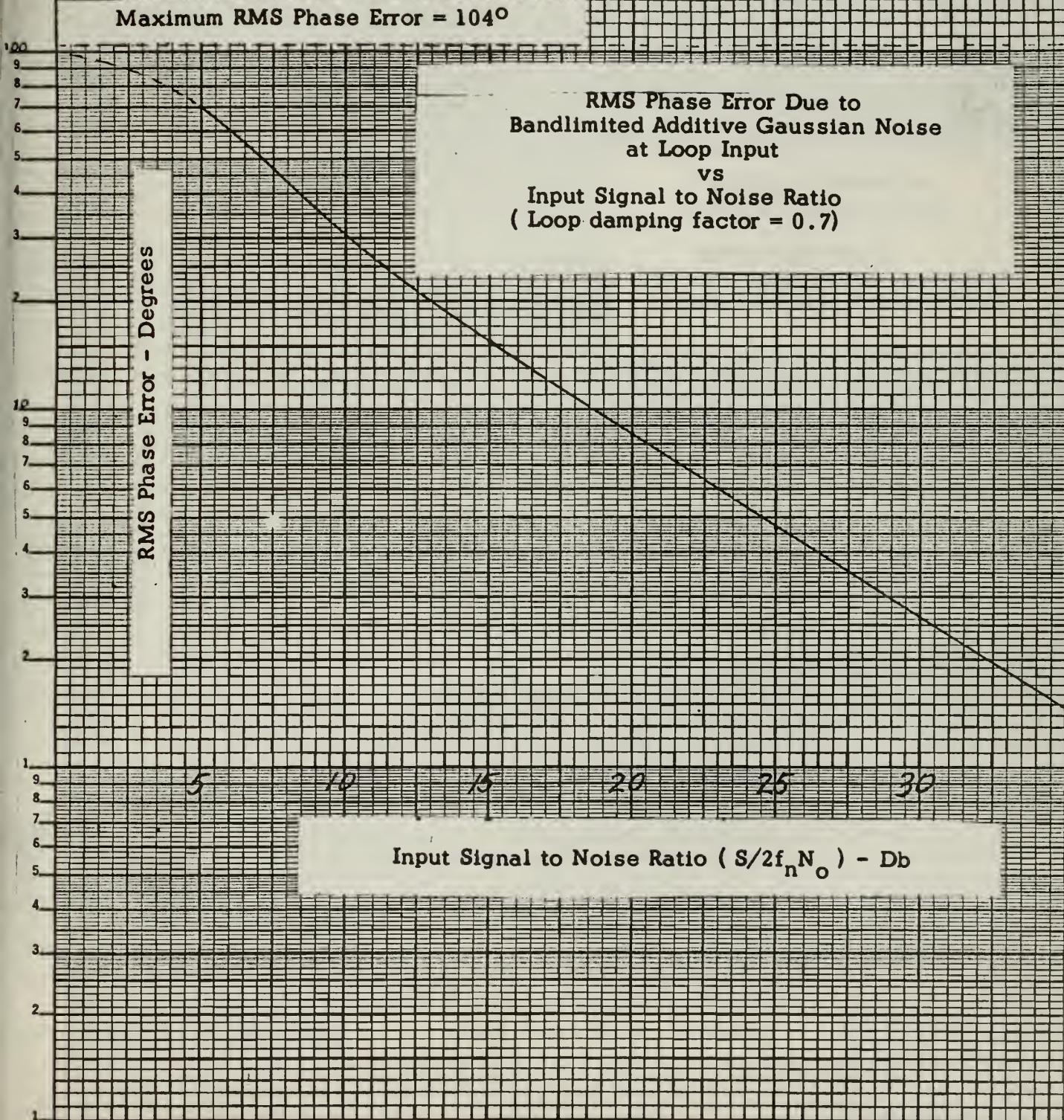




FIG. 5-3

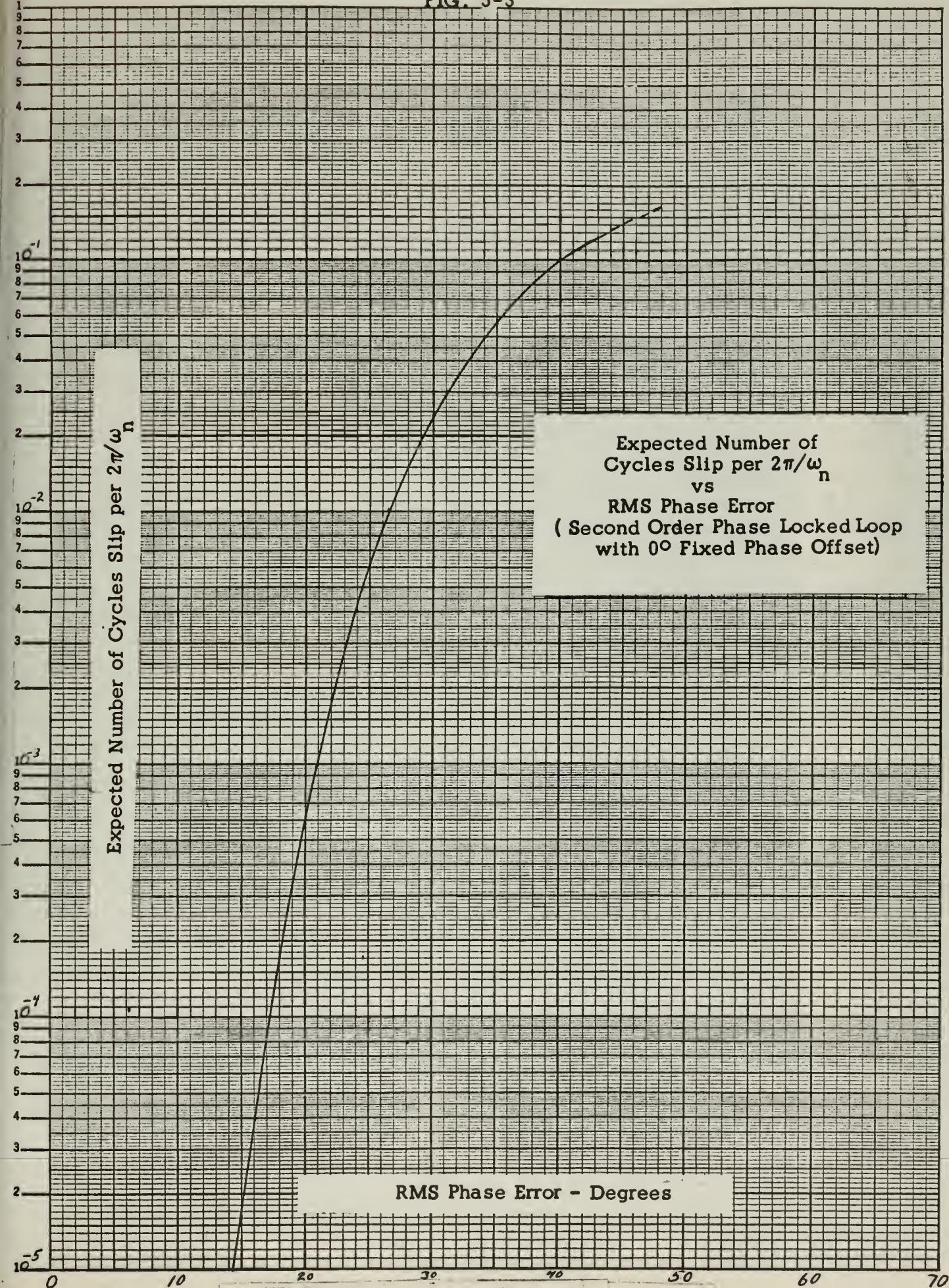




FIG. 5-4

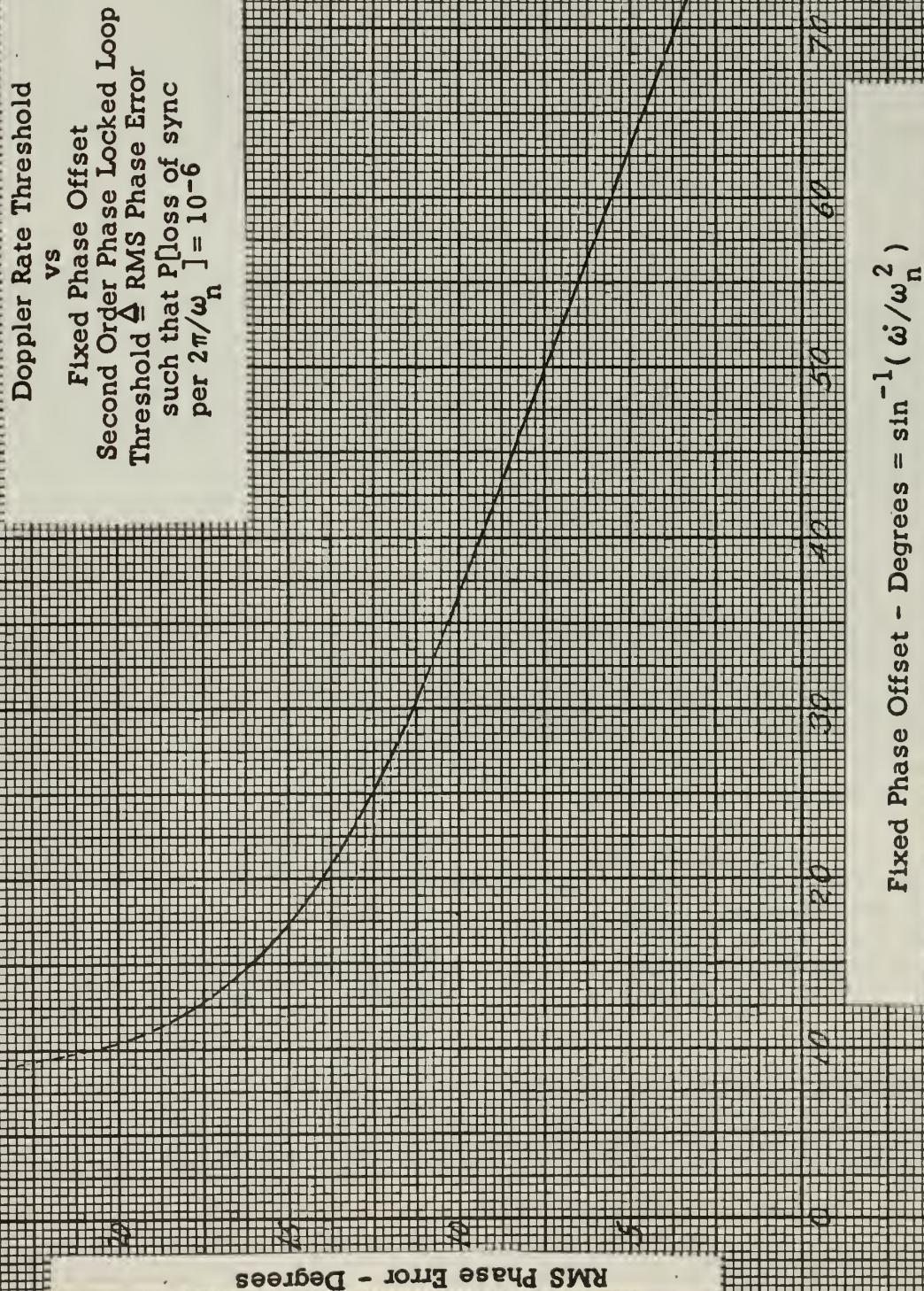




FIG. 5-5

Expected Additional Number of Cycles Slip per Peak of Phase Error

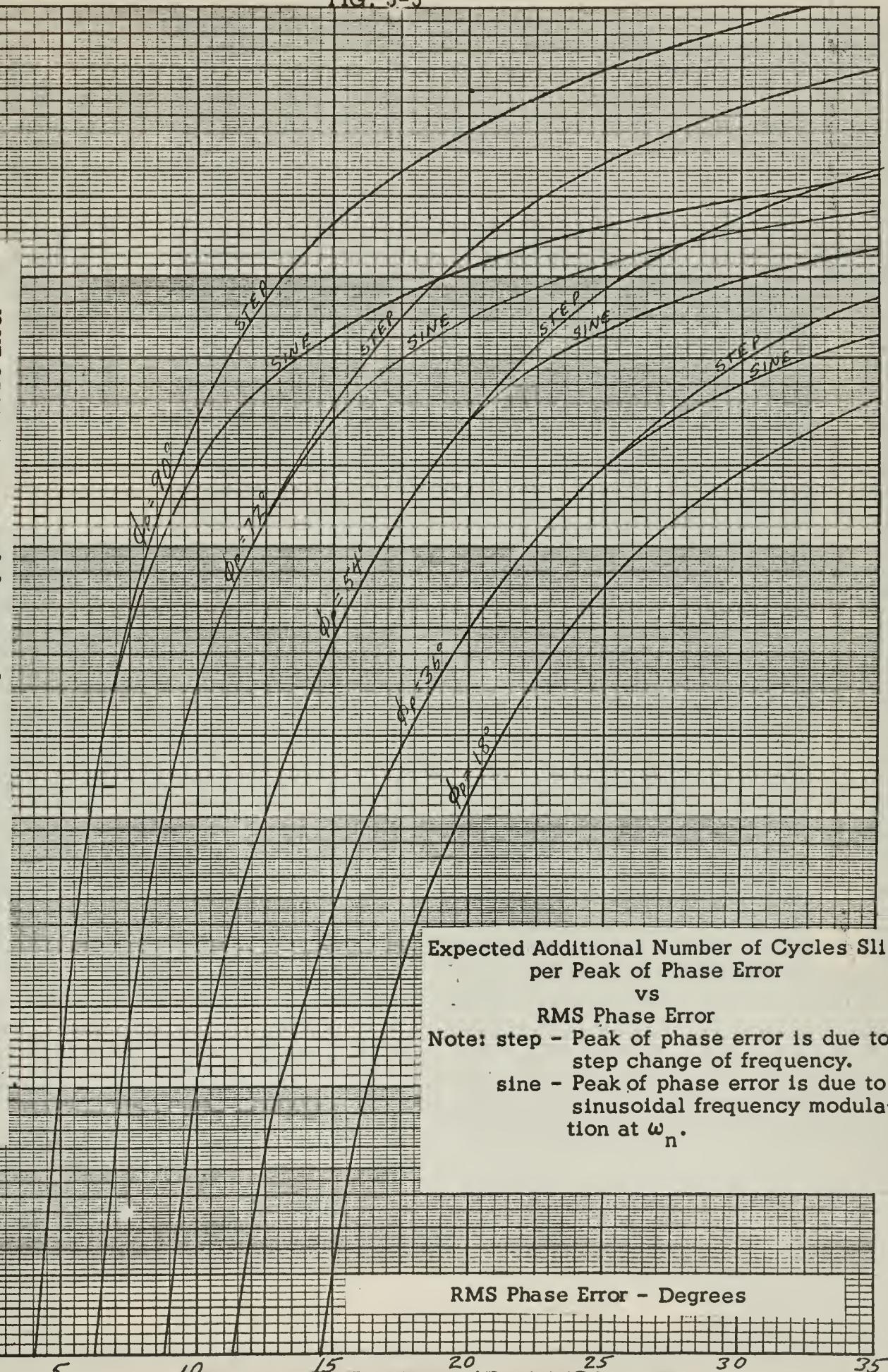
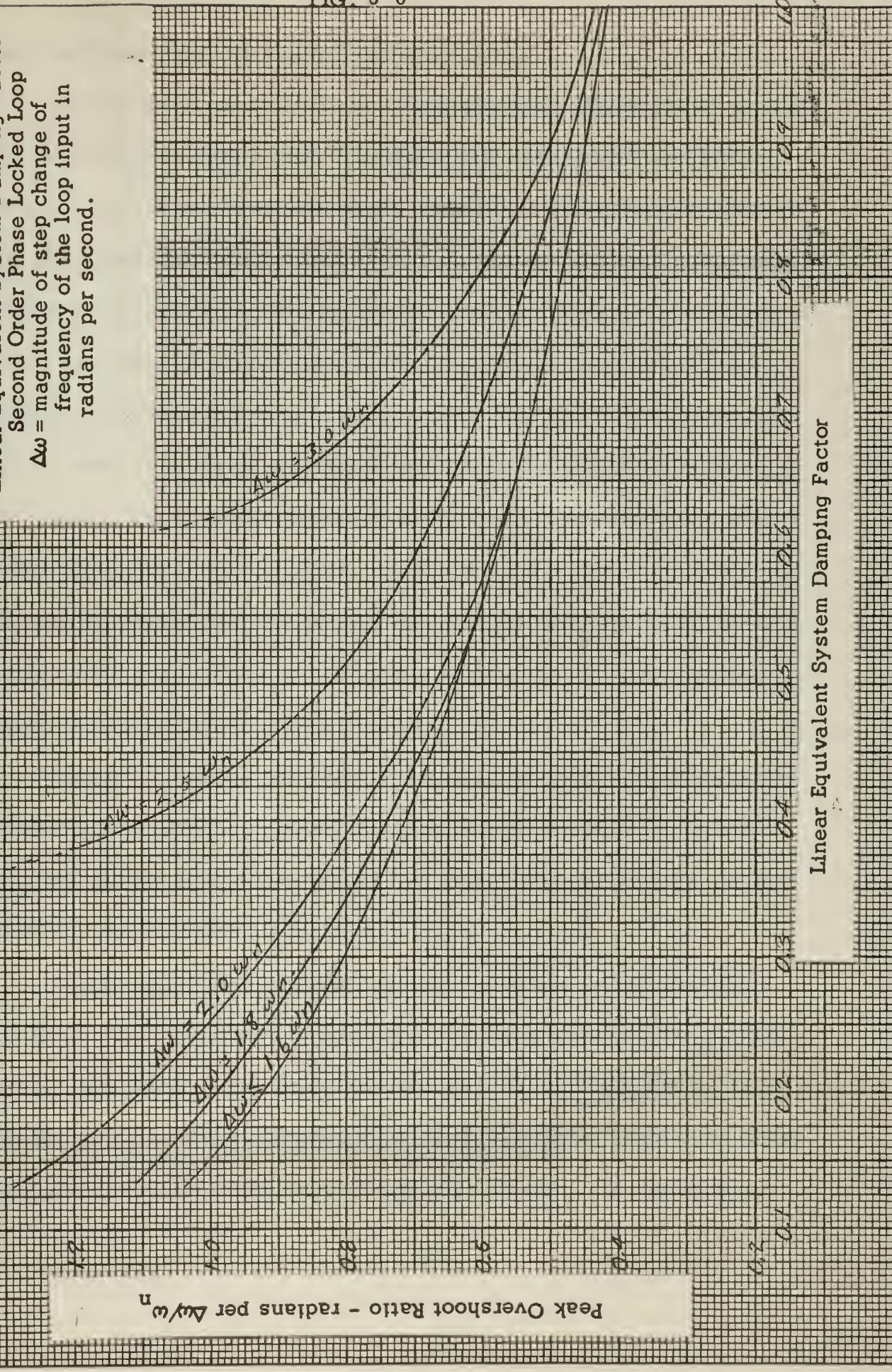




FIG. 5-6

Peak Overshoot Ratio - radians per  $\Delta\omega/\omega_n$

Linear Equivalent System Damping Factor  
 Second Order Phase Locked Loop  
 $\Delta\omega$  = magnitude of step change of  
 frequency of the loop input in  
 radians per second.





find peak phase error for a given frequency step.

Figures 5-7 through 5-13 are phase plane diagrams of phase locked loop performance. These show frequency error, phase error and phase plane trajectories under various signal modulation and damping conditions. The trajectories go counter clockwise about the stable point, rather than clockwise as is generally accepted and is done in figure 2-5. Figure 5-10 shows the effect of cycle slipping. This is without noise and is due entirely to the peak phase error being greater than  $180^\circ$ . The conditions are the same as in figure 5-9 except that the peak phase error is only  $135^\circ$  there. The transient generated in the phase comparator output due to the cycle slip should be noted. It can be seen in figure 5-10 that the transient bandwidth is much greater than the bandwidth of the phase error signal in the loop, and so may be easily filtered out of the phase comparator output and used to drive a counter.

Figure 5-11 shows the effect of extreme underdamping of the phase locked loop. Figure 5-12 shows that with a ramp of frequency for the input the second order phase locked loop has a fixed phase error. It may be shown that this phase error is  $\sin^{-1} \left( \frac{\dot{\omega}}{\omega_n} \right)$ .

Figure 5-13 shows the effect of noise on phase locked loop performance. The top figure is the phase plane trajectory when there is no noise at the input. When noise is added as in the second picture, note that the variance of the phase error increases rapidly at large phase errors. Note also that some noise peaks are causing slipping. In the third pic-



ture the magnitude of the step of frequency has been reduced slightly, which causes considerable reduction of the peak phase error. There is much less non-linear operation of the loop and no noticeable cycle slipping.

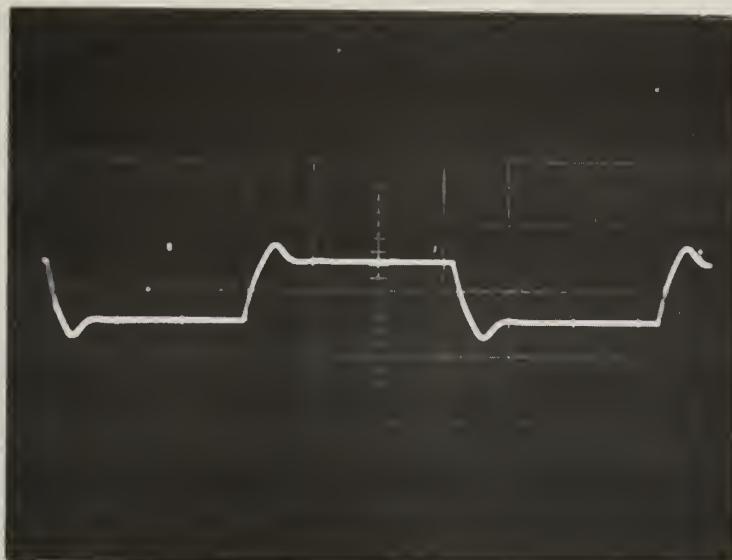
This equipment set-up has allowed a detailed study of phase locked loop threshold effects and dynamic performance. This was necessary in order that sufficient information on loop performance could be gained to formulate the definitions of optimum phase locked loop and threshold.



FIG. 7  
PHASE-LOCKED LOOP PERFORMANCE  
damping factor = 1.002

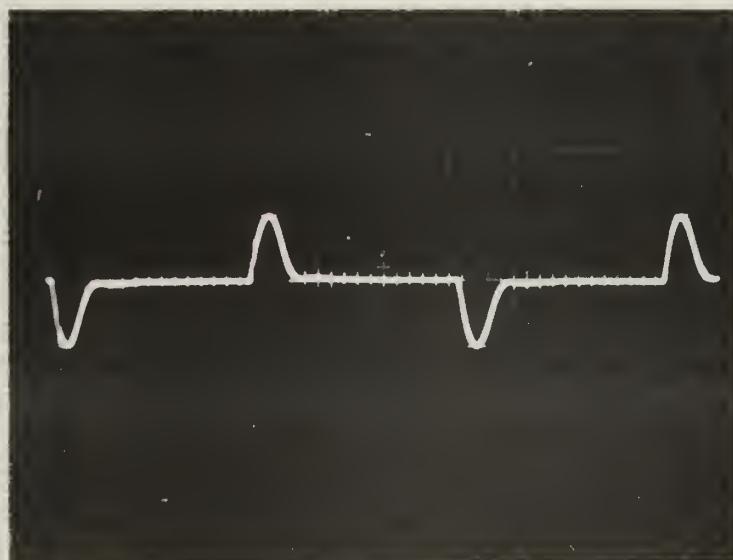
$\omega_n = 475$  cps

$\Delta\omega = 3 \omega_n$



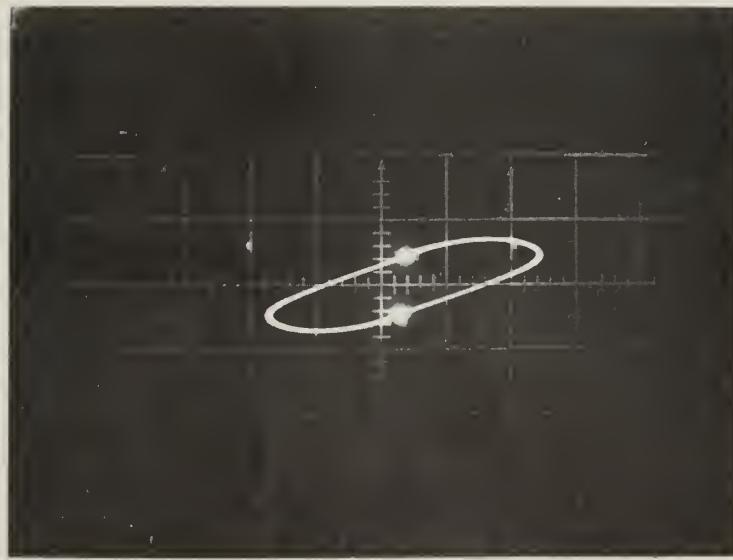
Frequency vs. Time

Horiz. 2 ms/cm  
Vert.  $2.5 \omega_n$ /cm



Phase vs. Time

Horiz. 2 ms/cm  
Vert.  $90^\circ$ /cm



Phase Plane

Horiz.  $45^\circ$ /cm  
Vert.  $2.5 \omega_n$ /cm



FIG. 8  
PHASE-LOCKED LOOP PERFORMANCE  
damping factor = 0.696

$\omega_n = 475$  cps

Freq. step =  $2 \omega_n$

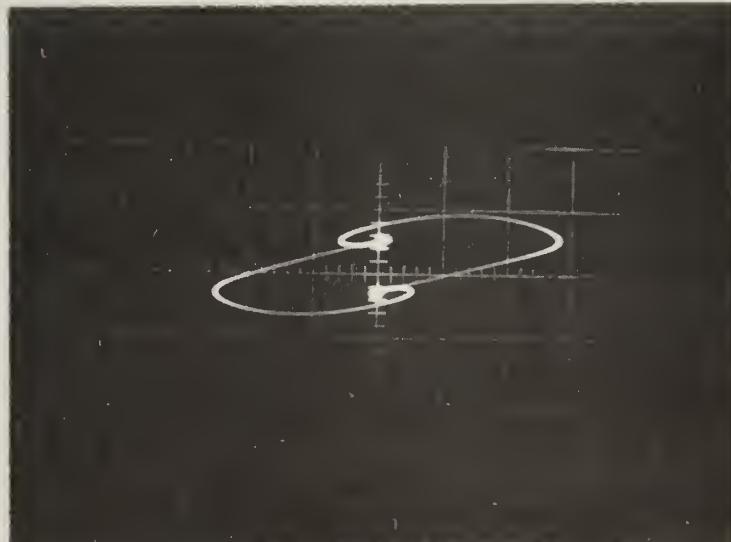
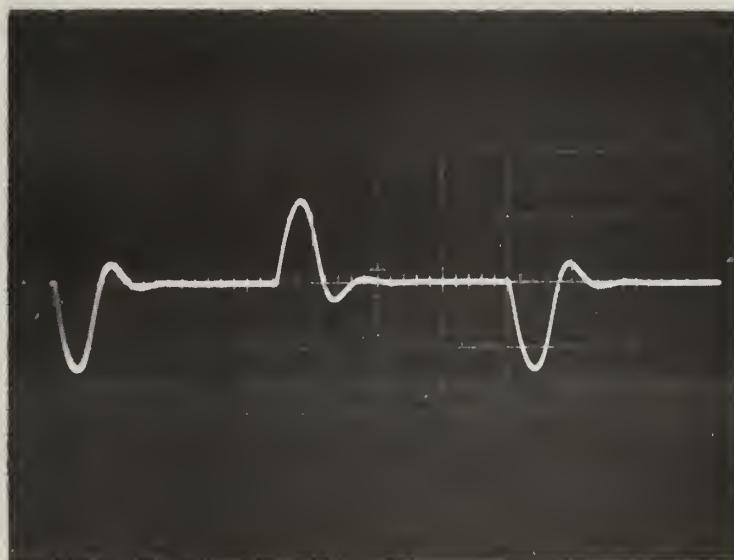
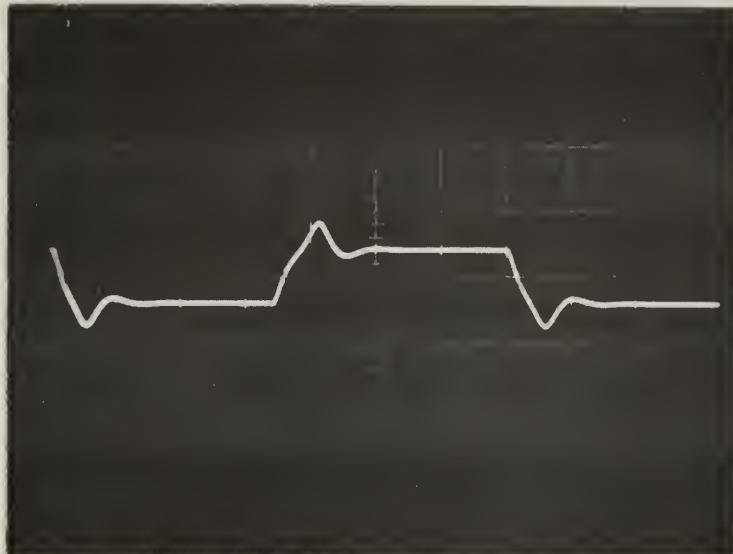
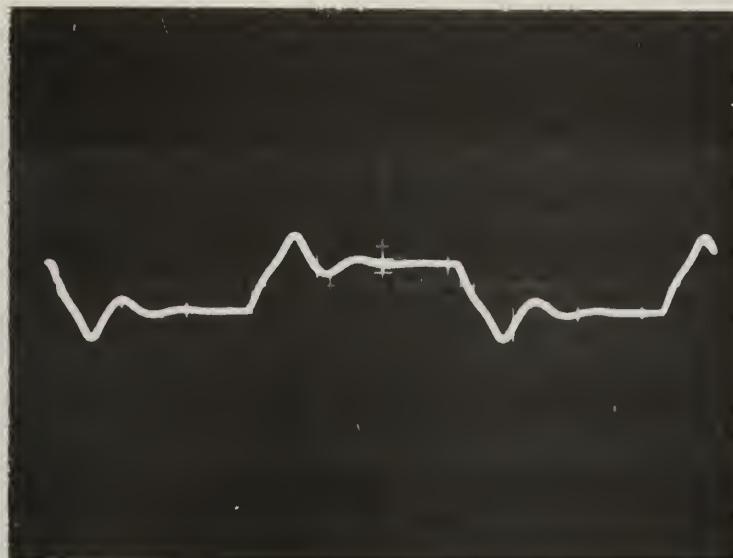




FIG. 9  
PHASE-LOCKED LOOP PERFORMANCE  
damping factor = 0.487

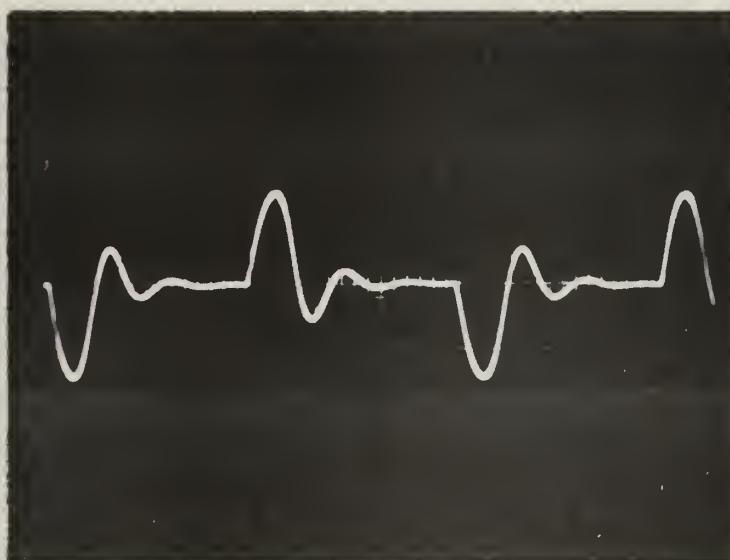
$\omega_n = 475$  cps

$\Delta\omega = 2 \omega_n$



Frequency vs. Time

Horiz. 2 ms/cm  
Vert. 2.5  $\omega_n$ /cm



Phase vs. Time

Horiz. 2 ms/cm  
Vert. 90°/cm



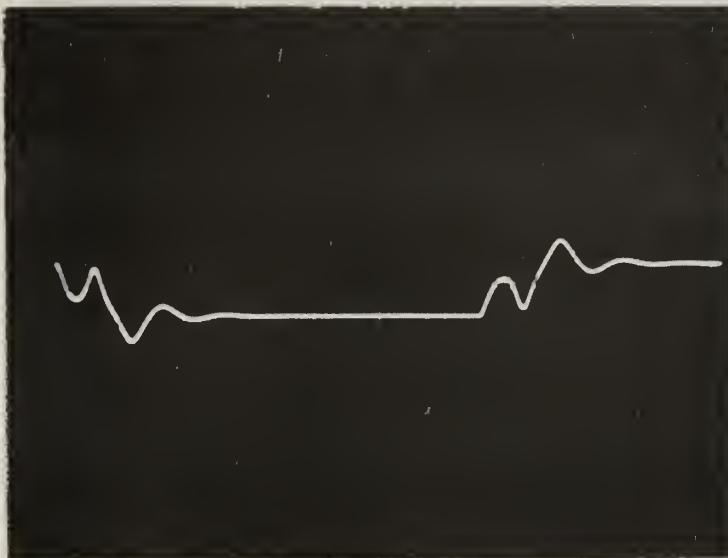
Phase Plane

Horiz. 45°/cm  
Vert. 2.5  $\omega_n$ /cm



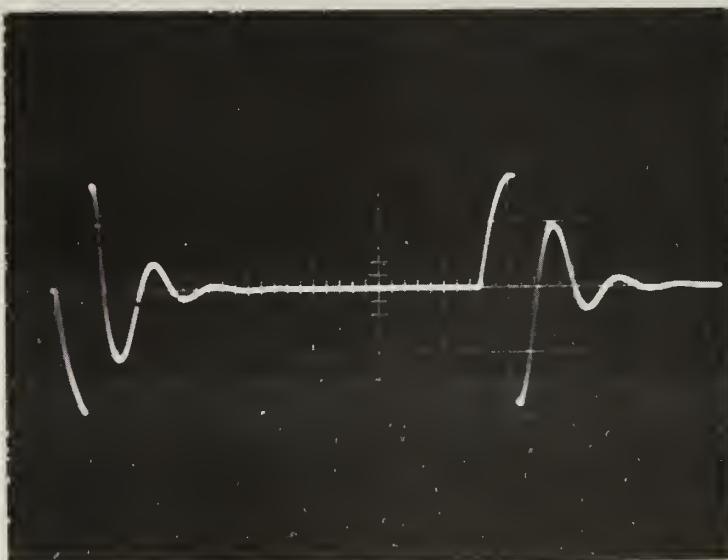
FIG. 10  
PHASE-LOCKED LOOP PERFORMANCE  
damping factor = 0.487

$\Delta\omega = 3 \omega_n$



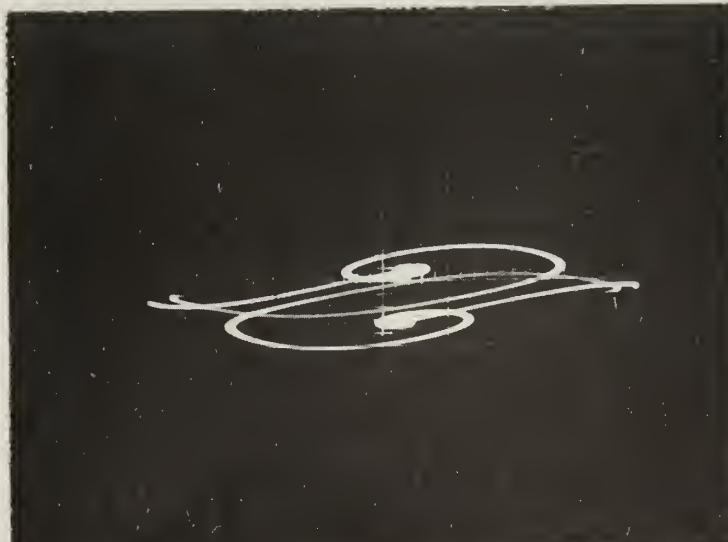
Frequency vs. Time

Horiz. 2 ms/cm  
Vert.  $2.5 \omega_n$ /cm



Phase vs. Time

Horiz. 2 ms/cm  
Vert.  $90^\circ$ /cm



Phase Plane

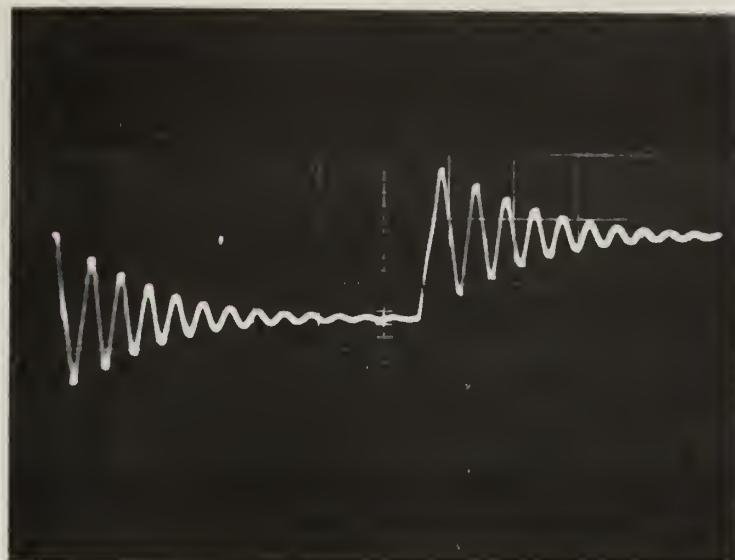
Horiz.  $45^\circ$ /cm  
Vert.  $2.5 \omega_n$ /cm



FIG. 11  
PHASE-LOCKED LOOP PERFORMANCE  
damping factor = 0.148

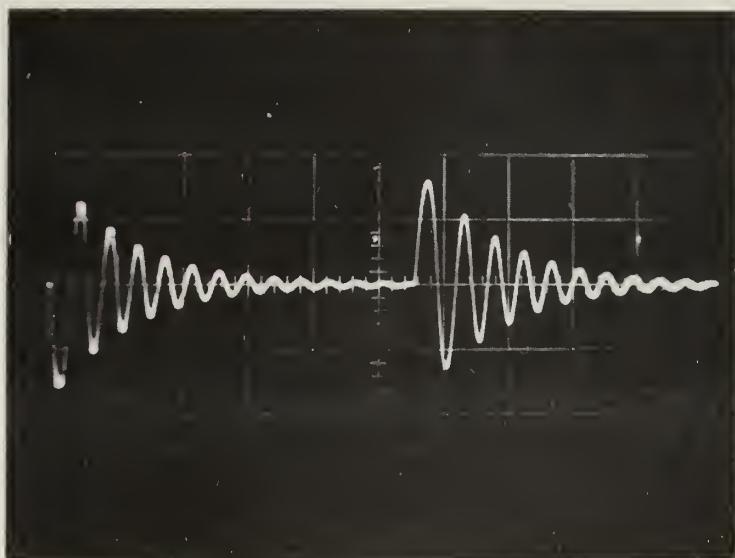
$\omega_n = 475$  cps

$\Delta\omega = \omega_n$



Frequency vs. Time

Horiz. 5 ms/cm  
Vert.  $0.6 \omega_n$ /cm



Phase vs. Time

Horiz. 5 ms/cm  
Vert.  $90^\circ/\text{cm}$



Phase Plane

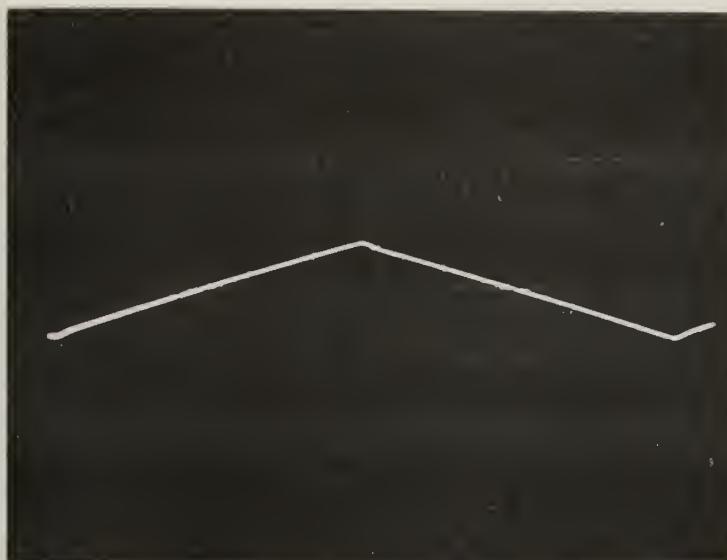
Horiz.  $45^\circ/\text{cm}$   
Vert.  $0.6 \omega_n/\text{cm}$



FIG. 12  
PHASE LOCKED LOOP PERFORMANCE  
damping factor = 0.696

$\tau_n = 475$  cps

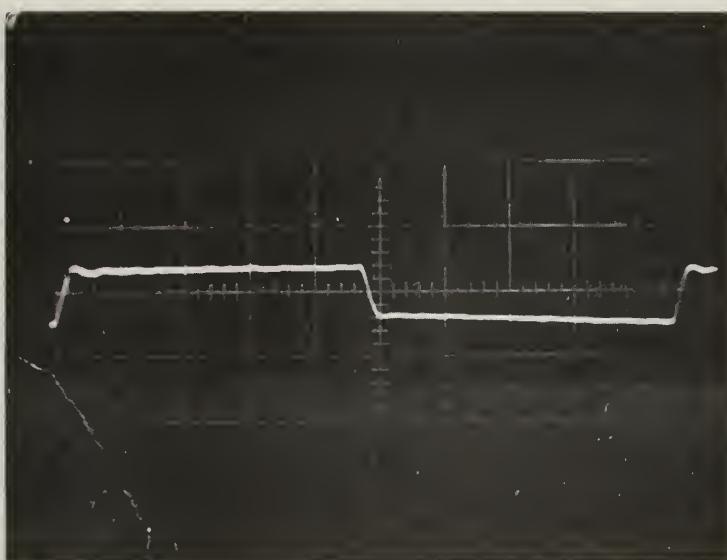
$$\frac{\omega^*}{\omega_n^2} = 0.122$$



Frequency vs. Time

Horiz. 2 ms/cm

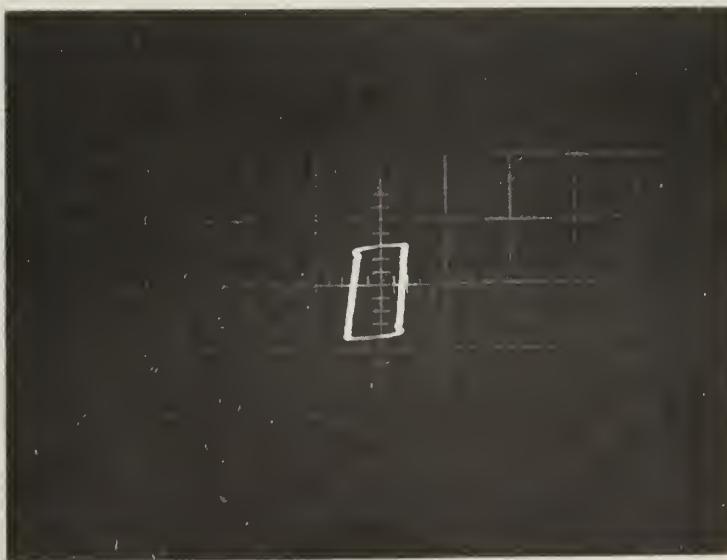
Vert. 2.0  $\tau_n$ /cm



Phase vs. Time

Horiz. 2 ms/cm

Vert. 18°/cm



Phase Plane

Horiz. 18°/cm

Vert. 2  $\omega_n$ /cm



FIG. 13  
PHASE LOCKED LOOP PERFORMANCE IN NOISE  
damping factor = 0.696

$\omega_n = 475$  cps



Phase Plane

Horiz.  $45^\circ/\text{cm}$   
Vert.  $1.458 \omega_n/\text{cm}$

$\Delta\omega = 3.0 \omega_n$

$\varphi_p = 130^\circ$

$\varphi_e = 0$

$S/N > 50$  db



Phase Plane

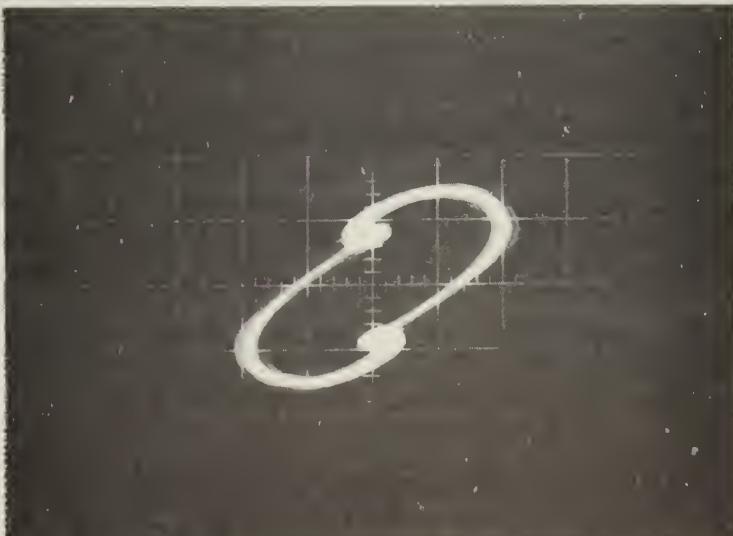
Horiz.  $45^\circ/\text{cm}$   
Vert.  $1.458 \omega_n/\text{cm}$

$\Delta\omega = 3.0 \omega_n$

$\varphi_p = 130^\circ$

$\varphi_e = 3.15^\circ$

$\frac{S}{2f_N} = 27.6$  db  
n.o



Phase Plane

Horiz.  $45^\circ/\text{cm}$   
Vert.  $1.458 \omega_n/\text{cm}$

$\Delta\omega = 2.5 \omega_n$

$\varphi_p = 90^\circ$

$\varphi_e = 3.15^\circ$

$\frac{S}{2f_N} = 27.6$  db  
n.o



## 6. Conclusions:

The problem of design of an optimum phase locked loop has been approached by first defining "optimum." This definition was made with the thought of maximizing that quality of a phase locked loop which makes it useful in most applications - the ability to operate in noise levels much greater than is possible with incoherent detectors. To do this it was also necessary to define the threshold of a phase locked loop in such a way that the application in which it is to be used enters the definition.

The result is a method of determination of the optimum values of all the design parameters of the loop from empirical data, eliminating guess work and prototype models which must be tested and adjusted to some optimum.

It may be possible to use an approximate mathematical model such as proposed by Lawhorn (2) which will predict the expected number of cycles slip in various conditions. This would eliminate the need for design data curves, if the model can be shown to be sufficiently accurate.

The data presented here is for the design of a phase locked loop having a bandpass filter at the input and an integral plus proportional loop filter. Further data are needed on other combinations of input and loop filters.

If this data on other loop conditions supports the accuracy of prediction which was obtained with the author's data, it is believed that this threshold definition and optimum design criterion will find extensive usefulness in all future phase locked loop design problems.



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## APPENDIX A

### A Derivation of Optimum Parameters

#### for the Linearized Phase Locked Loop

This derivation will find the optimum phase locked loop parameters for a step change of frequency at the loop input, with bandpass gaussian noise added. The criterion for optimization will be to minimize the sum of the mean square error due to transient peaks and the variance of the receiver's estimate of  $\theta(t)$  due to noise at the input.

The input signal, in Laplace notation is:

$$\theta_1(s) + \theta_n(s) = \frac{\Delta\omega}{s^2} + \theta_n(s)$$

where  $\Delta\omega$  is the magnitude of the step in radians per second.

The power spectral density of  $\theta_n(s)$  is from Sec. 2:

$$G_{\theta_n}(\omega) = \frac{2N_0}{A^2}$$

The function to be minimized is:

$$\int_0^\infty e^2(t)dt \left| \begin{array}{l} \theta_n(s)=0 \\ \theta_1(s)=\frac{\Delta\omega}{s^2} \end{array} \right. + \int_0^\infty \hat{\theta}^2(t)dt \left| \begin{array}{l} \theta_n(s) \\ \theta_1(s)=0 \end{array} \right.$$

The minimization will be carried out utilizing Liapunov's second method. The transfer function of the loop is as given in Section two. These must be rewritten in state space notation.



$$\frac{e(s)}{\theta_1(s)} = \frac{s^2}{s^2 + K\tau s + K}$$

Which is in equivalent state space notation:

$$\dot{\underline{e}} = F\underline{e} + \underline{u}$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K & -K\tau \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta\omega,$$

or  $\dot{e}_1 = e_2$

$$\dot{e}_2 = -Ke_1 -K\tau e_2 + \Delta\omega.$$

We may now write in vector notation:

$$\int_0^\infty e^2(t)dt = \int_0^\infty \underline{e}^t Q \underline{e} dt \text{ where } Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

but this integral, by Liapunov is:

$$\int_0^\infty \underline{e}^t Q \underline{e} dt = e^t(0) P e(0) = V_e.$$

Where  $P$  is defined by:

$$-Q = F^t P + P F.$$

If the inputs are imbedded, then in Laplace notation:

$$e(s) (s^2 + K\tau s + K) - s e(0) - \dot{e}(0) - e(0) = s^2 \frac{\Delta\omega}{s^2},$$

and  $e(0) = 0$

$$\dot{e}(0) = \Delta\omega$$

The  $P$  matrix is found as follows:

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -K \\ 1 & -K\tau \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -K & -K\tau \end{bmatrix}$$



$$\therefore P = \begin{bmatrix} (\tau/2) + 1/2K\tau & 1/2K \\ 1/2K & 1/2K^2\tau \end{bmatrix}$$

Now  $V_e$  may be found:

$$V_e = \underline{e}^t(0) P \underline{e}(0) = \frac{\Delta\omega^2}{2K^2\tau}$$

Similarly for the noise input:

$$\int_0^\infty \hat{\theta}^2(t) dt = \hat{\theta}^t(0) P \hat{\theta}(0) = V_{\hat{\theta}}$$

and imbedding inputs:

$$\theta(s)(s^2 + sK\tau + K) = \theta_n(s)sK\tau + \theta_n(s)K + \theta(0) + s\theta(0) + K\tau\theta(0)$$

$$\hat{\theta}(0) = K\tau\theta_n(s); \quad \dot{\hat{\theta}}(0) = (K - K^2\tau^2)\theta_n(s)$$

Now  $V_{\hat{\theta}}$  may be found:

$$V_{\hat{\theta}} = \hat{\theta}^t(0) P \hat{\theta}(0) = \theta_n(s) \left[ \frac{K\tau}{2} + 1/2\tau \right]$$

Averaging  $\theta_n(s)$  over the ensemble of all possible noise waveforms gives the power spectral density of the noise.

$$V_{\hat{\theta}} = \frac{2N_0}{A^2} \left[ K\tau + 1/2\tau \right]$$

Now minimize  $V_e + V_{\hat{\theta}}$  with respect to  $K$  and  $\tau$ :

$$V = V_e + V_{\hat{\theta}} = \frac{\Delta\omega^2}{2K^2\tau} + \frac{N_0K\tau}{A^2} + \frac{N_0}{A^2\tau}$$

$$\frac{\partial V}{\partial K} = \frac{-\Delta\omega^2}{K^3\tau} + \frac{N_0\tau}{A^2} = 0$$

$$\frac{\partial V}{\partial \tau} = \frac{-\Delta\omega^2}{2K^2\tau^2} + \frac{N_0K}{A^2} - \frac{N_0}{A^2\tau^2} = 0$$

Solving simultaneously for  $K$  and  $\tau$ :



$$K = \frac{\Delta \omega A}{\sqrt{2} N_0} = \Delta \omega \sqrt{\frac{S}{N_0}}$$

$$\tau = \sqrt{\frac{1}{K}} \sqrt{2}$$

The optimum values for bandwidth and damping factor may now be found:

$$\omega_n = \sqrt{K} = \sqrt{\Delta \omega} \left( \frac{S}{N_0} \right)^{\frac{1}{4}}$$

$$\zeta = \frac{\tau}{2\sqrt{K}} = \frac{1}{2}\sqrt{2} = 0.7$$

These values of  $\omega_n$  and  $\zeta$  are optimum by this method, and agree with the work of Jaffe and Rechtin (7). However this derivation was done to show a method of optimizing the linearized phase locked loop, not to prove the validity of the values derived. To optimize under more general criteria the  $Q$  matrix may be changed, the quantities minimized may be changed, and the  $F$  matrix may be changed. Changing the  $F$  matrix allows adding a cost function to the state space variables which may be a very complex function of those variables. The result is a method of optimization which is quite general.







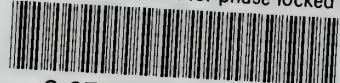






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A threshold criterion for phase locked I



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